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Physical variational principle and thin plate theory in electro-magneto-elastic analysis

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ABSTRACT

Under the assumption of the quasi-static electric and magnetic fields the electro-magneto-elastic analysis including medium and its environment is studied in this paper. The complete governing equations under the finite deformation can be derived from the physical variational principle. In the physical variational principle the variations of the electric potential and magnetic potential are divided into local variations and migratory variations. From the virtual change of the sum of the electromagnetic energy and the couple energy produced by the migratory variation we can get the electromagnetic force and in this case the virtual variation of the volume should be considered. It is also found that the Maxwell stress is directly related to the strain in a material with piezoelectric or piezomagnetic behavior for the finite deformation case. The thin plate theory in first order is derived from the general theory in this paper and the Maxwell stress is naturally included in the governing equations.

1. Introduction

The electro-magneto-elastic analyses in elastic electromagnetic media under the quasi-static electric and magnetic fields are very important due to its extensive applications in the engineering structures. In the previous literatures various theories were presented. Landau and Lifshitz (1960) and Stratton (1941) gave the derivation of the Maxwell stress in electroelastic media from the energy principle, but their formulas were not very clear and did not give an entire variational principle. Moon (1984) systematically discussed the magnetoelastic problem. Many important fundamental theories and formulas for the electro-magneto-elastic dynamics in finite deformation were established by Maugin (1988), Eringen and Maugin (1989), Pao (1978) and others. Some differences between these theories, especially in the electromagnetic force, were discussed in Kuang (2008a), Kuang (2008b), Kuang (2009a) Jiang and Kuang (2004), Jiang and Kuang (2006), Zhou and Zheng (1997) and others. Some variational principles of electroelastic and magnetoelastic materials were discussed by Toupin (1956), Bustamante et al. (2008), etc. In our papers (2008a,2008b,2009a,2009b) we showed that together with the first law of thermodynamics, the known facts show that the following physical variational principle is also held. When variations of variables in the following variational formula are independent each other we have

$$\int_V \delta\Phi\,\mathrm{d}V = \delta W + \delta Q, \quad \text{or} \quad \int_V \delta\Phi\,\mathrm{d}V - \delta W - \delta Q = 0 \tag{1}$$

where δ is the variation sign, Φ is the internal energy per volume, W is the work done by the external force and electromagnetic field, Q is the heat supplied by the external heat source. In general W and Q are not the state functions, so they are expressed in differential forms under volume integrals. In this paper we only discuss the isothermal reversible case with $\delta Q = 0$, $\delta T = 0$. The physical variational principle with using the electromagnetic Gibbs free energy may be the most important one which will be discussed in this paper. Let g be the electromagnetic Gibbs free energy per volume:

$$\delta g - \delta w^* - \delta Q = 0, \quad g = \Phi - Ts - E_i D_i - B_i H_i \tag{2}$$

where w^* is the sum of the work of the external force on the medium and the complementary work of the medium on the electromagnetic field. T and s are the temperature and the entropy per volume respectively, $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, $\mathbf{B}$ are the electric field strength, electric displacement, magnetic field strength and magnetic induction density respectively. Under the quasi-static or static electric and magnetic fields we have

$$\mathbf{E} = -\nabla\varphi, \quad \mathbf{H} = -\nabla\psi \tag{3}$$

where φ is the electric potential and ψ is the magnetic scalar potential.

In this paper we discuss the physical variational principle and Maxwell stress in electro-magneto-elastic analysis under finite deformation. In our previous papers (2008b,2009a) we assumed that the finite strain is still not too large, so we can neglect the influence of strains on the Maxwell stress. But when the finite strains cannot be neglected, we shall show that the Maxwell stress is related to strains when the initial configuration is taken as a reference configuration.

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In this paper we also slightly modify the physical variational principle (Kuang, 2008b) in a more exact form. It is shown that in the physical variational principle the variations of the electric potential and magnetic potential are divided into local variations and migratory variations. The local variations of φ and ψ are induced by changes of φ and ψ themselves. The migratory variations of φ and ψ are induced by changes of φ and ψ due to virtual displacements. From the virtual change of the sum of the electromagnetic energy and the couple energy produced by the migratory variation we can get the electromagnetic force by the conservation law of the energy and in this case the virtual variation of the volume should be considered. Using the physical variational principle we can simply derive the governing equations of electro-magneto-elastic analysis and the Maxwell stress is obtained naturally.

The thin plate theory in first order which is somewhat ambiguous (Zhou and Zheng, 1997) in literatures is derived from the general theory under the small deformation. This theory is fully reasonable and the Maxwell stress is also naturally included in the governing equation.

2. The electromagnetic Gibbs free energy in the initial configuration

2.1. Some fundamental formulas in finite deformation

Fundamental formulas in finite deformation can be found in many books (Kuang, 2002; Ogden, 1984). Here we give some fundamental formulas for use in this paper. In this paper, notations $\rho,\ \rho_e,\ \mathbf{u},\ \sigma,\ \boldsymbol{\varepsilon},\ \mathbf{D},\ \mathbf{E},\ \mathbf{B},\ \mathbf{H}$ and $\varphi,\ \psi$ will denote the density, electric charge density, displacement, stress, strain, electric displacement, electric field, magnetic induction, magnetic field and electric potential, magnetic scalar potential respectively in the current configuration. The physical quantities in the initial configuration are expressed by an upper bar '–' on the corresponding physical quantities, such as $\bar{\rho},\ \bar{\rho}_e,\ \bar{\mathbf{u}},\ \bar{\boldsymbol{\sigma}},\ \bar{\boldsymbol{\varepsilon}},\ \overline{\mathbf{D}},\ \overline{\mathbf{E}},\ \overline{\mathbf{B}},\ \overline{\mathbf{H}}$ and $\bar{\varphi},\ \bar{\psi}$, where $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$ are the second kind of the Piola–Kirchhoff stress and Green strain, respectively. The coordinate and the subscripts in the current configuration are denoted by small letters, such as x_i, while the coordinate and the subscripts in the initial configuration are denoted by capital letters, such as $X_I,\ \bar{\sigma}_{IJ}$. It should be noted that the differentiation with the capital or small letter subscript is different, such as $x_{i,J} = \partial x_i/\partial X_J,\ u_{i,J} = u_{i,m}x_{m,J}$, etc. Because in this paper the same coordinate system in the current and initial configurations is taken, we also have $X_I = X_i,\ \overline{\sigma}_{IJ} = \overline{\sigma}_{ij}$, etc. when there is no differential symbol. The transformation relations of the mechanical variables in the current and initial configurations are

$$
\begin{aligned}
&\bar{\rho}_e = j\rho_e,\ \ \bar{\rho} = j\rho,\ \ \mathrm{d}V = j\,\mathrm{d}\overline{V},\ \ \bar{f}_K = jf_k,\ \ \overline{\Phi} = j\Phi,\ \ \bar{g} = jg,\\
&j = |x_{i,J}|\\
&\bar{\sigma}_{IJ} = jX_{I,m}X_{J,n}\sigma_{mn},\quad \bar{n}_K\,\mathrm{d}\bar{a} = Jx_{i,K}n_i\,\mathrm{d}a,\quad \bar{\sigma}^*\,\mathrm{d}\bar{a} = \sigma^*\,\mathrm{d}a,\quad J = j^{-1}\\
&\overline{T}_{Kl}\bar{n}_K\,\mathrm{d}\bar{a} = T_{kl}n_k\,\mathrm{d}a,\quad \overline{T}_k = jX_{J,n}\bar{n}_J n_n T_k,\quad T_i = Jx_{l,K}n_l N_K\overline{T}_i\\
&x_i = X_I + u_i,\quad x_{i,J}X_{J,l} = \delta_{il}\bar{\varepsilon}_{IJ} = (x_{k,I}x_{k,J} - \delta_{IJ})/2,\quad \delta\bar{\varepsilon}_{IJ} = x_{k,I}\delta u_{k,J}
\end{aligned}
\tag{4}
$$

The variational principle in elasticity under finite deformation is

$$
\begin{aligned}
&\delta\overline{\Pi} = \int_V \delta\bar{g}\,\mathrm{d}V - \delta\overline{W}^* = 0,\\
&\delta\overline{W}^* = \int_{\overline{V}}(\bar{f}_k - \bar{\rho}\ddot{u}_k)\delta u_k\,\mathrm{d}\overline{V} + \int_{\bar{a}_\sigma}\overline{T}_k^*\delta u_k\,\mathrm{d}\bar{a}
\end{aligned}
\tag{5}
$$

The quantity $\delta\mathbf{u}$ is the vector variation of $\mathbf{u}$. It is emphasized that Eq. (5) is a local variational principle, i.e. all the displacement variables can be varied independently. It means that the momentum equation is obtained from the local variational principle.

The electric displacement $\overline{D}_I$ and the magnetic induction $\overline{B}_I$ are defined by the principle that the electric charge and "magnetic charge (=0)" do not change with deformation in a closed surface consisting of the same material points, and the electric field $\overline{E}_I$ and $\overline{H}_I$ are defined by the derivative of φ and ψ with x_I, respectively (Eringen and Maugin, 1989; Kuang, 2002). So we have

$$
\begin{aligned}
&\oint_{\bar{a}}\overline{D}_{I,I}\,\mathrm{d}\bar{a} = \oint_a D_{i,i}\,\mathrm{d}a,\quad \overline{D}_I = jX_{I,m}D_m,\quad D_m = Jx_{m,I}\overline{D}_I\\
&\overline{B}_I = jX_{I,m}B_m,\quad B_m = Jx_{m,I}\overline{B}_I,\quad \sigma^*\,\mathrm{d}a = \bar{\sigma}^*\,\mathrm{d}\bar{a}\\
&\overline{E}_I = -\varphi_{,I} = -\varphi_{,m}x_{m,I} = x_{m,I}E_m,\quad E_m = X_{I,m}\overline{E}_I\\
&\overline{H}_I = -\psi_{,I} = x_{m,I}E_m,\quad H_m = X_{I,m}\overline{H}_I
\end{aligned}
\tag{6}
$$

From Eq. (6) it is known that $\overline{E}_{I,J} = \overline{E}_{J,I} = -\varphi_{,IJ}$ and $\overline{H}_{I,J} = \overline{H}_{J,I} = -\psi_{,IJ}$.

2.2. The electromagnetic Gibbs free energy in the initial configuration

Since the isothermal electromagnetic Gibbs free energy must be invariant in a rigid body rotation, the $\bar{g}$ for materials with symmetric material coefficients shown in Eq. (7) in the finite deformation problem should be in the following form

$$
\begin{aligned}
&\bar{g} = (1/2)\overline{C}_{IJKL}\bar{\varepsilon}_{JI}\bar{\varepsilon}_{LK} - (1/2)\bar{\epsilon}_{KL}\overline{E}_K\overline{E}_L - \bar{e}^e_{KIJ}\overline{E}_K\bar{\varepsilon}_{IJ} - (1/2)\bar{l}^e_{IJKL}\overline{E}_I\overline{E}_J\bar{\varepsilon}_{LK}\\
&\qquad - (1/2)\bar{\mu}_{KL}\overline{H}_K\overline{H}_L - \bar{e}^m_{KIJ}\overline{H}_K\bar{\varepsilon}_{IJ} - (1/2)\bar{l}^m_{IJKL}\overline{H}_I\overline{H}_J\bar{\varepsilon}_{LK} + \cdots\\
&\overline{C}_{IJKL} = \overline{C}_{JIKL} = \overline{C}_{IJLK} = \overline{C}_{KLIJ},\quad \bar{\epsilon}_{KL} = \bar{\epsilon}_{LK},\\
&\bar{e}^e_{KIJ} = \bar{e}^e_{KJI},\quad \bar{\mu}_{KL} = \bar{\mu}_{LK},\\
&\bar{e}^m_{KIJ} = \bar{e}^m_{KJI},\quad \bar{l}^e_{IJKL} = \bar{l}^e_{JIKL} = \bar{l}^e_{IJLK} = \bar{l}^e_{KLIJ},\quad \bar{l}^m_{IJKL} = \bar{l}^m_{JIKL} = \bar{l}^m_{IJLK} = \bar{l}^m_{KLIJ}
\end{aligned}
\tag{7}
$$

where $\overline{C}_{IJKL},\ \bar{\epsilon}_{KL},\ \bar{e}_{KIJ},\ \bar{l}^e_{IJKL},\ \bar{\mu}_{KL},\ \bar{e}^m_{KIJ},\ \bar{l}^m_{IJKL}$ are the material coefficients. It is noted that coefficients with a upper bar "–" in the initial configuration are different with coefficients without bar in the current configuration, but there are relations between them. From the thermodynamic theory, the constitutive equations are

$$
\begin{aligned}
&\bar{\sigma}_{KL} = \partial\bar{g}/\partial\bar{\varepsilon}_{LK} = \overline{C}_{IJKL}\bar{\varepsilon}_{IJ} - \bar{e}^e_{JKL}\overline{E}_J\\
&\qquad - (1/2)\bar{l}^e_{IJKL}\overline{E}_I\overline{E}_J - \bar{e}^m_{JKL}\overline{H}_J - (1/2)\bar{l}^m_{IJKL}\overline{H}_I\overline{H}_J\\
&\overline{D}_K = -\partial\bar{g}/\partial\overline{E}_K = \tilde{\epsilon}^e_{KL}E_L + \bar{e}^e_{KIJ}\varepsilon_{IJ},\quad \tilde{\epsilon}^e_{KL} = \bar{\epsilon}_{KL} + \bar{l}^e_{IJKL}\varepsilon_{JI}\\
&\overline{B}_K = -\partial\bar{g}/\partial\overline{H}_K = \tilde{\epsilon}^m_{KL}H_L + \bar{e}^m_{KIJ}\varepsilon_{IJ},\quad \tilde{\epsilon}^m_{KL} = \bar{\mu}_{KL} + \bar{l}^m_{IJKL}\varepsilon_{JI}
\end{aligned}
\tag{8}
$$

Using Eq. (8), Eq. (7) is reduced to

$$
\begin{aligned}
&\bar{g} = (1/2)\overline{C}_{IJKL}\bar{\varepsilon}_{JI}\bar{\varepsilon}_{LK} + \bar{g}^{em},\quad \bar{g}^{em} = -(1/2)(\overline{D}_K\overline{E}_K + \overline{B}_K\overline{H}_K + \overline{\Gamma}_{KL}\bar{\varepsilon}_{LK})\\
&\overline{\Gamma}_{KL} = \bar{e}^e_{MKL}\overline{E}_M + \bar{e}^m_{MKL}\overline{H}_M = -\bar{e}^e_{MKL}\varphi_{,M} - \bar{e}^m_{MKL}\psi_{,M}
\end{aligned}
\tag{9}
$$

In $\bar{g}$ the term $(1/2)\overline{C}_{IJKL}\bar{\varepsilon}_{JI}\bar{\varepsilon}_{LK}$ is the mechanical deformation energy, $(1/2)(\overline{D}_K\overline{E}_K + \overline{B}_K\overline{H}_K)$ is the electromagnetic energy, $\overline{\Gamma}_{KL0}\bar{\varepsilon}_{LK}$ is the mechanical and electromagnetic coupling energy, $\bar{g}^{em}$ is the sum of the electromagnetic energy and coupling energy. For the small deformation case $\overline{\Gamma}_{KL}\bar{\varepsilon}_{LK}$ can be neglected, so the coupling energy can also be neglected.

3. Variational principle under finite deformation

3.1. Standard variational principle under finite deformation

The key and difficult point of the physical variational principle in nonlinear electro-magneto-elastic media is that the virtual displacement can produce the variation of electric potential and the magnetic scalar potential. In the following sections we shall call the variations $\delta_u\varphi,\ \delta_u\psi,\ \delta_u\overline{\mathbf{E}},\ \delta_u\overline{\mathbf{H}}$ the migratory variations of $\varphi,\ \psi,\ \overline{\mathbf{E}},\ \overline{\mathbf{H}}$ produced by the virtual displacement vector $\delta\mathbf{u}$. So the variation of the virtual electric potential φ is divided into local variation $\delta_\varphi\varphi$ due to the variation of φ itself and migratory variation $\delta_u\varphi$ due to the variation of $\mathbf{u}$. Similarly for $\psi,\ \mathbf{E},\ \mathbf{H}$. So we have

$$
\begin{aligned}
&\delta\varphi = \delta_\varphi\varphi + \delta_u\varphi, \quad \delta\psi = \delta_\psi\psi + \delta_u\psi, \quad \delta\overline{E}_i = \delta_\varphi\overline{E}_i + \delta_u\overline{E}_i,\\
&\delta\overline{H}_i = \delta_\psi\overline{H}_i + \delta_u\overline{H}_i\\
&\delta_u\varphi = \varphi_{,p}\delta u_p = -E_p\delta u_p = -\overline{E}_L X_{L,p}\delta u_p, \quad \delta_u\psi = \psi_{,p}\delta u_p = -\overline{H}_L X_{L,p}\delta u_p\\
&\delta_u\overline{E}_I = -\delta_u\left(\varphi_{,I}\right) = -\varphi_{,Ip}\delta u_p = E_{p,I}\delta u_p = \overline{E}_{I,p}\delta u_p = \overline{E}_{I,L}X_{L,p}\delta u_p\\
&\qquad = \overline{E}_{L,I}X_{L,p}\delta u_p\\
&\delta_u\overline{H}_I = H_{p,I}\delta u_p = \overline{H}_{I,p}\delta u_p = \overline{H}_{L,I}X_{L,p}\delta u_p
\end{aligned}
\tag{10}
$$

The variational principle in electro-magneto-elastic media is somewhat different with that in elastic media, because in the electro-magneto-elastic variational principle migratory variations of electric potential and magnetic potential should be considered. The migratory variations of electric potential and magnetic potential are related to the electromagnetic force. From the energy conservative principle the variation of the electromagnetic energy and the coupling energy produced by the migratory variations is equal to the work done by the effective electromagnetic force. Considering these reasons the standard variational principle under finite deformation for electro-magneto-elastic media can be expressed in the following form.

Let the displacement vector $\mathbf{u}$, the electric potential φ and the magnetic potential ψ satisfy their boundary conditions on their own boundaries $\bar{a}_u,\ \bar{a}_\varphi,\ \bar{a}_\psi$, and the continuity conditions on their interface $\bar{a}^{\mathrm{int}}$ (Fig. 1). The standard electromagnetic Gibbs free energy variational principle under finite deformation can be expressed as

$$
\begin{aligned}
&\delta\overline{\Pi} = \delta\overline{\Pi}_1 + \delta\overline{\Pi}_2 - \delta\overline{W}^{int*} = 0\\
&\delta\overline{\Pi}_1 = \int_{\overline{V}}\delta\bar{g}\,\mathrm{d}\overline{V} + \int_{\overline{V}}\bar{g}^{em}\delta u_{i,i}\,\mathrm{d}\overline{V} - \delta\overline{W}^*\\
&\delta\overline{\Pi}_2 = \int_{\overline{V}^{env}}\delta\bar{g}^{env}\,\mathrm{d}\overline{V} + \int_{\overline{V}^{env}}\bar{g}^{envem}\delta u^{env}_{i,i}\,\mathrm{d}\overline{V} - \delta\overline{W}^{env*}\\
&\delta\overline{W}^* = \int_{\overline{V}}(\bar{f}_k - \bar{\rho}\ddot{u}_k)\delta u_k\,\mathrm{d}\overline{V} - \int_{\overline{V}}\bar{\rho}_e\delta\varphi\,\mathrm{d}\overline{V} + \int_{\bar{a}_\sigma}\overline{T}^*_k\delta u_k\,\mathrm{d}\bar{a}\\
&\qquad - \int_{\bar{a}_q}\bar{\sigma}^*\delta\varphi\,\mathrm{d}\bar{a} + \int_{\bar{a}^{env}_\mu}\overline{B}^*_I\bar{n}_I\delta\psi\,\mathrm{d}\bar{a}\\
&\delta\overline{W}^{*env} = \int_{\overline{V}^{env}}(\bar{f}^{env}_k - \bar{\rho}\ddot{u}^{env}_k)\delta u^{env}_k\,\mathrm{d}\overline{V} - \int_{\overline{V}^{env}}\bar{\rho}^{env}_e\delta\varphi^{env}\,\mathrm{d}\overline{V}\\
&\qquad + \int_{\bar{a}^{env}_\sigma}\overline{T}^{env*}_k\delta u^{env}_k\,\mathrm{d}\bar{a} - \int_{\bar{a}^{env}_q}\bar{\sigma}^{env*}\delta\varphi^{env}\,\mathrm{d}\bar{a}\\
&\qquad + \int_{\bar{a}^{env}_\mu}\overline{B}^{env*}_I\bar{n}_I\delta\psi^{env}\,\mathrm{d}\bar{a}\\
&\delta\overline{W}^{int*} = \int_{\bar{a}^{int}}\overline{T}^{int*}_k\delta u_k\,\mathrm{d}\bar{a} - \int_{\bar{a}^{int}}\bar{\sigma}^{int*}\delta\varphi\,\mathrm{d}\bar{a} + \int_{\bar{a}^{int}_\mu}\overline{B}^{int*}_I\bar{n}_I\delta\psi^{env}\,\mathrm{d}\bar{a}
\end{aligned}
\tag{11}
$$

where $\overline{T}^*_k,\ \bar{\sigma}^*,\ \overline{B}^*_I;\ \overline{T}^{env*}_k,\ \bar{\sigma}^{env*},\ \overline{B}^{env*}_I;\ \overline{T}^{int*}_k,\ \bar{\sigma}^{int*},\ \overline{B}^{int*}_I$ are the given values on the corresponding surfaces. In Eq. (11) we use $\int_{\overline{V}}\delta\bar{g}\,\mathrm{d}\overline{V} + \int_{\overline{V}}\bar{g}^{em}\delta u_{i,i}\,\mathrm{d}\overline{V}$ instead of $\delta\int_V g\,\mathrm{d}V$ in previous paper (Kuang, 2008b) in order to be appropriate for large finite strains.

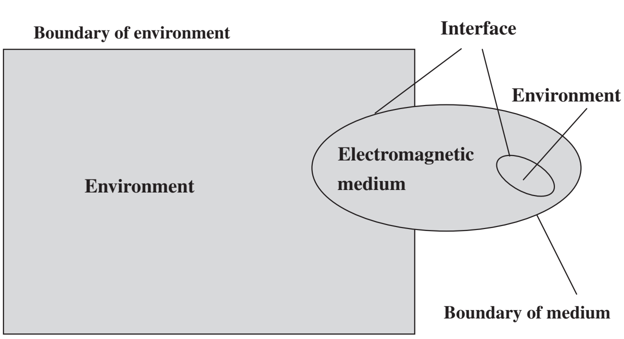

Fig. 1. Electromagnetic medium and its environment.

Using Eq. (10) and $\delta u_{k,k} = \delta u_{k,J}X_{J,k}$, we have

$$
\begin{aligned}
&\int_{\overline{V}}\delta\bar{g}\,\mathrm{d}\overline{V} + \int_{\overline{V}}\bar{g}^{em}\delta u_{k,k}\,\mathrm{d}\overline{V} = \int_V\delta g\,\mathrm{d}V + \int_V g^{em}\delta u_{k,k}\,\mathrm{d}V\\
&= \int_{\overline{V}}\bar{\sigma}_{IJ}\delta\bar{\varepsilon}_{IJ}\,\mathrm{d}\overline{V} - \int_{\overline{V}}\overline{D}_I\delta\overline{E}_I\,\mathrm{d}\overline{V} - \int_{\overline{V}}\overline{B}_I\delta\overline{H}_I\,\mathrm{d}\overline{V}\\
&- \int_{\overline{V}}(1/2)\left(\overline{D}_k\overline{E}_k + \overline{B}_k\overline{H}_k + \overline{\Gamma}_{KL}\bar{\varepsilon}_{LK}\right)\delta u_{k,k}\,\mathrm{d}\overline{V}\\
&= \int_{\bar{a}}\left[\bar{\sigma}_{IJ}x_{k,I} - (1/2)\left(\overline{D}_K\overline{E}_K + \overline{B}_K\overline{H}_K + \overline{\Gamma}_{KL}\bar{\varepsilon}_{LK}\right)X_{J,k}\right]\bar{n}_J\delta u_k\,\mathrm{d}\bar{a}\\
&- \int_{\overline{V}}\left[\bar{\sigma}_{IJ}x_{k,I} - (1/2)\left(\overline{D}_K\overline{E}_K + \overline{B}_K\overline{H}_K + \overline{\Gamma}_{KL}\bar{\varepsilon}_{LK}\right)X_{J,k}\right]_{,J}\delta u_k\,\mathrm{d}\overline{V}\\
&+ \int_{\bar{a}}\overline{D}_I\bar{n}_I\delta_\varphi\varphi\,\mathrm{d}\bar{a} - \int_{\overline{V}}\overline{D}_{I,I}\delta_\varphi\varphi\,\mathrm{d}\overline{V} - \int_{\overline{V}}\overline{D}_I\overline{E}_{L,I}X_{L,p}\delta u_p\,\mathrm{d}\overline{V}\\
&+ \int_{\bar{a}}\overline{B}_I\bar{n}_I\delta_\psi\psi\,\mathrm{d}\bar{a} - \int_{\overline{V}}\overline{B}_{I,I}\delta_\psi\psi\,\mathrm{d}\overline{V} - \int_{\overline{V}}\overline{B}_I\overline{H}_{L,I}X_{L,p}\delta u_p\,\mathrm{d}\overline{V}
\end{aligned}
\tag{12}
$$

Therefore $\delta\overline{\Pi}_1$ in Eq. (11) is reduced to

$$
\begin{aligned}
&\delta\overline{\Pi}_1 = \delta\overline{\Pi}'_1 + \delta\overline{\Pi}''_1\\
&\delta\overline{\Pi}'_1 = \int_{\bar{a}_\sigma}\left(\bar{\sigma}_{IJ}x_{k,I}\bar{n}_J - \overline{T}^*_k\right)\delta u_k\,\mathrm{d}\bar{a} + \int_{\bar{a}^{int}}\bar{\sigma}_{IJ}x_{k,I}\bar{n}_J\delta u_k\,\mathrm{d}\bar{a}\\
&\quad - \int_{\overline{V}}\left\{\left(\bar{\sigma}_{IJ}x_{k,I}\right)_{,J} + \bar{f}_k - \bar{\rho}\ddot{u}_k\right\}\delta u_k\,\mathrm{d}\overline{V} - \int_{\overline{V}}\left(\overline{D}_{I,I} - \bar{\rho}_e\right)\delta_\varphi\varphi\,\mathrm{d}\overline{V}\\
&\quad + \int_{\bar{a}_q}\left(\overline{D}_I\bar{n}_I + \bar{\sigma}^*\right)\delta_\varphi\varphi\,\mathrm{d}\bar{a} + \int_{\bar{a}^{int}}\overline{D}_I\bar{n}_I\delta_\varphi\varphi\,\mathrm{d}\bar{a} - \int_{\overline{V}}\overline{B}_{I,I}\delta_\psi\psi\,\mathrm{d}\overline{V}\\
&\quad + \int_{\bar{a}_\mu}\left(\overline{B}_I - \overline{B}^*_I\right)\bar{n}_I\delta_\psi\psi\,\mathrm{d}\bar{a} + \int_{\bar{a}^{int}}\overline{B}_I\bar{n}_I\delta_\psi\psi\,\mathrm{d}\bar{a}\\
&\delta\overline{\Pi}''_1 = (1/2)\int_{\bar{a}_\sigma}\left[\overline{D}_N\varphi_{,N} + \overline{B}_N\psi_{,N} + \overline{\Gamma}_{KL}\bar{\varepsilon}_{LK}\right]X_{J,k}\bar{n}_J\delta u_k\,\mathrm{d}\bar{a}\\
&\quad + (1/2)\int_{\bar{a}^{int}}\left[\overline{D}_N\varphi_{,N} + \overline{B}_N\psi_{,N} + \overline{\Gamma}_{KL}\bar{\varepsilon}_{LK}\right]X_{J,k}\bar{n}_J\delta u_k\,\mathrm{d}\bar{a}\\
&\quad - (1/2)\int_{\overline{V}}\left\{\left[\overline{D}_N\varphi_{,N} + \overline{B}_N\psi_{,N} + \overline{\Gamma}_{KL}\bar{\varepsilon}_{LK}\right]X_{J,k}\right\}_{,J}\delta u_k\,\mathrm{d}\overline{V}\\
&\quad - \int_{\overline{V}}\overline{D}_I\overline{E}_{L,I}X_{L,p}\delta u_p\,\mathrm{d}\overline{V} - \int_{\overline{V}}\bar{\rho}_e\overline{E}_p\delta u_p\,\mathrm{d}\overline{V} - \int_{\bar{a}_q}\bar{\sigma}^*\overline{E}_p\delta u_p\,\mathrm{d}\bar{a}\\
&\quad - \int_{\overline{V}}\overline{B}_I\overline{H}_{L,I}X_{L,p}\delta u_p\,\mathrm{d}\overline{V} + \int_{\bar{a}^{env}_\mu}\overline{B}^*_I\bar{n}_I\overline{H}_p\delta u_p\,\mathrm{d}\bar{a}
\end{aligned}
\tag{13}
$$

where $\delta\overline{\Pi}'_1$ is the part of $\delta\overline{\Pi}_1$ due to the local variations of $\mathbf{u},\ \varphi,\ \psi$; $\delta\overline{\Pi}''_1$ is the part of $\delta\overline{\Pi}_1$ due to the migratory variations of $\varphi,\ \psi$. Substituting the following identity

$$
\begin{aligned}
&- \int_{\overline{V}}\overline{D}_I\overline{E}_{L,I}X_{L,p}\delta u_p\,\mathrm{d}\overline{V} - \int_{\overline{V}}\bar{\rho}_e E_p\delta u_p\,\mathrm{d}\overline{V} - \int_{\bar{a}_q}\bar{\sigma}^* E_p\delta u_p\,\mathrm{d}\bar{a}\\
&\quad - \int_{\overline{V}}\overline{B}_I\overline{H}_{L,I}X_{L,p}\delta u_p\,\mathrm{d}\overline{V} + \int_{\bar{a}_\mu}\overline{B}^*_I\bar{n}_I\overline{H}_p\delta u_p\,\mathrm{d}\bar{a} = -\int_{\bar{a}_q}\left(\overline{D}_I\bar{n}_I + \bar{\sigma}^*\right)E_p\delta u_p\,\mathrm{d}\bar{a}\\
&\quad - \int_{\bar{a}^{int}}\overline{D}_I\bar{n}_I E_p\delta u_p\,\mathrm{d}\bar{a} + \int_{\overline{V}}\overline{D}_I\overline{E}_L X_{L,p}\delta u_{p,I}\,\mathrm{d}\overline{V} + \int_{\overline{V}}\left(\overline{D}_{I,I} - \bar{\rho}_e\right)E_p\delta u_p\,\mathrm{d}\overline{V}\\
&\quad - \int_{\bar{a}_\mu}\left(\overline{B}_I - \overline{B}^*_I\right)\bar{n}_I H_p\delta u_p\,\mathrm{d}\bar{a} - \int_{\bar{a}^{int}}\overline{B}_I\bar{n}_I H_p\delta u_p\,\mathrm{d}\bar{a} + \int_{\overline{V}}\overline{B}_I\overline{H}_L X_{L,p}\delta u_{p,I}\,\mathrm{d}\overline{V}\\
&\quad + \int_{\overline{V}}\overline{B}_{I,I}H_p\delta u_p\,\mathrm{d}\overline{V} = \int_{\bar{a}_q}\left(\overline{D}_I\bar{n}_I + \bar{\sigma}^*\right)\delta_u\varphi\,\mathrm{d}\bar{a} + \int_{\bar{a}^{int}}\overline{D}_I\bar{n}_I\delta_u\varphi\,\mathrm{d}\bar{a}\\
&\quad + \int_{\bar{a}}\left(\overline{D}_I\overline{E}_L X_{L,p}\right)\bar{n}_I\delta u_p\,\mathrm{d}\bar{a} - \int_{\overline{V}}\left(\overline{D}_I\overline{E}_L X_{L,p}\right)_{,I}\delta u_p\,\mathrm{d}\overline{V} - \int_{\overline{V}}\left(\overline{D}_{I,I} - \bar{\rho}_e\right)\delta_u\varphi\,\mathrm{d}\overline{V}\\
&\quad + \int_{\bar{a}_\mu}\left(\overline{B}_I - \overline{B}^*_I\right)\bar{n}_I\delta_u\psi\,\mathrm{d}\bar{a} + \int_{\bar{a}^{int}}\overline{B}_I\bar{n}_I\delta_u\psi\,\mathrm{d}\bar{a} + \int_{\bar{a}}\left(\overline{B}_I\overline{H}_L X_{L,p}\right)\bar{n}_I\delta u_p\,\mathrm{d}\bar{a}\\
&\quad - \int_{\overline{V}}\left(\overline{B}_I\overline{H}_L X_{L,p}\right)_{,I}\delta u_p\,\mathrm{d}\overline{V} - \int_{\overline{V}}\overline{B}_{I,I}\delta_u\psi\,\mathrm{d}\overline{V}
\end{aligned}
\tag{14}
$$

and the relation $X_{L,p}\delta u_p\overline{E}_L = -X_{L,p}\delta u_p\varphi_{,L} = -\delta u_p\varphi_{,p} = E_p\delta u_p$ into $\delta\overline{\Pi}''_1$ in Eq. (13) we get

$$\begin{aligned}\delta\overline{\Pi}_1'' = &-\int_{\overline{V}}(\overline{D}_{I,I}-\bar{\rho}_e)\delta_u\varphi\,\mathrm{d}\overline{V}+\int_{\bar{a}_q}(\overline{D}_I\bar{n}_I+\bar{\sigma}^*)\delta_u\varphi\,\mathrm{d}\bar{a}+\int_{\bar{a}^{int}}\overline{D}_I\bar{n}_I\delta_u\varphi\,\mathrm{d}\bar{a}\\ &-\int_{\overline{V}}\overline{B}_{I,I}\delta_u\psi\,\mathrm{d}\overline{V}+\int_{\bar{a}_\mu}(\overline{B}_I-\overline{B}_I^*)\bar{n}_I\delta_u\psi\,\mathrm{d}\bar{a}+\int_{\bar{a}^{int}}\overline{B}_I\bar{n}_I\delta_u\psi\,\mathrm{d}\bar{a}\\ &+\int_{\bar{a}_\sigma}\bar{\sigma}^M_{IL}X_{L,k}\bar{n}_I\delta u_k\,\mathrm{d}\bar{a}+\int_{\bar{a}^{int}}\bar{\sigma}^M_{IL}X_{L,k}\bar{n}_I\delta u_k\,\mathrm{d}\bar{a}-\int_{\overline{V}}(\bar{\sigma}^M_{IL}X_{L,k})_{,I}\delta u_k\,\mathrm{d}\overline{V}\\ \bar{\sigma}^M_{IL} = &\,\overline{D}_I\overline{E}_L+\overline{B}_I\overline{H}_L-(1/2)\left(\overline{D}_N\overline{E}_N+\overline{B}_N\overline{H}_N+\overline{\Gamma}_{MN}\bar{\varepsilon}_{NM}\right)\delta_{IL}\end{aligned}\tag{15}$$

where $\bar{\sigma}^M_{IL}$ is the "second kind" of the Maxwell stress in the initial configuration. From Eqs. (13) and (15) it is found that the terms before $\delta_\varphi\varphi$, $\delta_u\varphi$ and $\delta\varphi=\delta_\varphi\varphi+\delta_u\varphi$ are the same, and so for $\delta_\psi\psi, \delta_u\psi$ and $\delta\psi$. Substituting Eq. (15) into (13), after some manipulation we get

$$\begin{aligned}\delta\overline{\Pi}_1 = &\int_{\bar{a}_\sigma}(\overline{S}_{Ij}\bar{n}_I-\overline{T}_j^*)\delta u_j\,\mathrm{d}\bar{a}+\int_{\bar{a}^{int}}\overline{S}_{Ij}\bar{n}_I\delta u_j\,\mathrm{d}\bar{a}-\int_{\overline{V}}(\overline{S}_{Ij,I}+\bar{f}_j-\bar{\rho}\ddot{u}_j)\delta u_j\,\mathrm{d}\overline{V}\\ &+\int_{\bar{a}_q}(\overline{D}_I\bar{n}_I+\bar{\sigma}^*)\delta\varphi\,\mathrm{d}\bar{a}+\int_{\bar{a}^{int}}\overline{D}_I\bar{n}_I\delta\varphi\,\mathrm{d}\bar{a}-\int_{\overline{V}}(\overline{D}_{I,I}-\bar{\rho}_e)\delta\varphi\,\mathrm{d}\overline{V}\\ &+\int_{\bar{a}_\mu}(\overline{B}_I\bar{n}_I-\overline{B}_I^*)\bar{n}_I\delta\psi\,\mathrm{d}\bar{a}+\int_{\bar{a}^{int}}\overline{B}_I\bar{n}_I\delta\psi\,\mathrm{d}\bar{a}-\int_{\overline{V}}\overline{B}_{I,I}\delta\psi\,\mathrm{d}\overline{V}\end{aligned}\tag{16}$$

where $\overline{S}_{Ij}$ is the pseudo-total stress (Jiang and Kuang, 2004), which is not the true stress in the dielectric and

$$\overline{S}_{Jk}=S_{Jk}+X_{L,k}\bar{\sigma}^M_{JL},\quad S_{Jk}=x_{k,I}\bar{\sigma}_{IJ}\tag{17}$$

It is well known that $S_{Jk}=x_{k,I}\bar{\sigma}_{IJ}$ is the first kind of the Piola–Kirchhoff stress (Kuang, 2002; Ogden, 1984), so that we call $X_{L,k}\bar{\sigma}^M_{JL}$ the "first kind" of the Maxwell stress in the initial configuration. Due to the arbitrariness of δu_i, $\delta\varphi$ and $\delta\psi$, from Eq. (16) we get

$$\begin{aligned}&\overline{S}_{Jk,J}+\bar{f}_k=\bar{\rho}\ddot{u}_k,\quad \overline{D}_{I,I}=\bar{\rho}_e,\quad \overline{B}_{I,I}=0,\quad \text{in } \overline{V}\\ &\overline{S}_{Jk}\bar{n}_J=\overline{T}_k^*,\quad \text{on } \bar{a}_\sigma;\quad \overline{D}_I\bar{n}_I=-\bar{\sigma}^*,\quad \text{on } \bar{a}_q;\quad \overline{B}_I=\overline{B}_I,\quad \text{on } \bar{a}_\mu\\ &\delta\overline{\Pi}_1=\int_{\bar{a}^{int}}\overline{S}_{Ij}\bar{n}_I\delta u_j\,\mathrm{d}\bar{a}+\int_{\bar{a}^{int}}\overline{D}_I\bar{n}_I\delta\varphi\,\mathrm{d}\bar{a}+\int_{\bar{a}^{int}}\overline{B}_I\bar{n}_I\delta\psi\,\mathrm{d}\bar{a}\end{aligned}\tag{18}$$

For the environment we have the similar formula:

$$\begin{aligned}&\overline{S}^{env}_{Ij,I}+\bar{f}^{env}_j=\bar{\rho}^{env}\ddot{u}^{env}_j,\quad \overline{D}^{env}_{I,I}=\bar{\rho}^{env}_e,\quad \overline{B}_{I,I}=0,\quad \text{in } \overline{V}^{env}\\ &\overline{S}^{env}_{Ij}\bar{n}^{env}_I=\overline{T}^{env*}_j,\quad \text{on } \bar{a}^{env}_\sigma;\overline{D}^{env}_I\bar{n}^{env}_I=-\bar{\sigma}^{env*},\quad \text{on } \bar{a}^{env}_q;\\ &\overline{B}^{env}_I=\overline{B}^{env*}_I,\quad \text{on } \bar{a}^{env}_\mu\\ &\delta\overline{\Pi}_2=\int_{\bar{a}^{int}}\overline{S}^{env}_{Ij}\bar{n}^{env}_I\delta u^{env}_j\,\mathrm{d}\bar{a}+\int_{\bar{a}^{int}}\overline{D}^{env}_I\bar{n}^{env}_I\delta\varphi^{env}\,\mathrm{d}\bar{a}\\ &\qquad+\int_{\bar{a}^{int}}\overline{B}^{env}_I\bar{n}^{env}_I\delta\psi^{env}\,\mathrm{d}\bar{a}\\ &\overline{S}^{env}_{Jk}=x_{k,I}\bar{\sigma}^{env}_{IJ}+X_{L,k}\bar{\sigma}^{envM}_{JL}\end{aligned}\tag{19}$$

Using $\bar{n}_I=-\bar{n}^{env}_I$, $u_I=u^{env}_I$, $\varphi=\varphi^{env}$ and $\delta\overline{\Pi}_1+\delta\overline{\Pi}_2=\delta\overline{W}^{int*}$ on the interface, we get

$$\begin{aligned}&(\overline{S}_{Ij}-\overline{S}^{env}_{Ij})\bar{n}_I=\overline{T}^{int*}_j,(\overline{D}_I-\overline{D}^{env}_I)\bar{n}_I=-\bar{\sigma}^{int*},\\ &\qquad(\overline{B}_I-\overline{B}^{env}_I)=\overline{B}^{int*}_I,\quad \text{on } a^{int}\end{aligned}\tag{20}$$

The above variational principle requests prior that the displacements, the electric potentials and the magnetic potentials satisfy their own boundary conditions and the continuity conditions on the interface, so the following equations should also be added to governing equations

$$\begin{aligned}&u_i=u_i^*,\quad \text{on } a_u;\quad \varphi=\varphi^*,\quad \text{on } a_\varphi;\quad \psi=0,\quad \text{on } a_\mu\\ &u_i^{env}=u_i^{*env},\quad \text{on } a_u^{env};\quad \varphi^{env}=\varphi^{*env};\quad \text{on } a_\varphi^{env};\quad \psi^{env}=0,\\ &\qquad \text{on } a_\mu^{env}\\ &u_i=u_i^{env},\quad \varphi=\varphi^{env},\quad \psi=\psi^{env};\quad \text{on } a^{int}\end{aligned}\tag{21}$$

Eqs. (17)–(21) are the governing equations under the finite deformation. It is noted that for the elastic material these formulas are reduced to the usual elastic governing equations for elasticity. When the deformation is not too large, then Eq. (11) is reduced to the formulas in previous previous paper (Kuang, 2008b) due to that terms $\overline{\Gamma}_{KL}\bar{\varepsilon}_{LK}$ and $\overline{C}_{IJKL}\bar{\varepsilon}_{JI}\bar{\varepsilon}_{LK}$ can be neglected and $\delta\int_V \bar{g}\,\mathrm{d}V=\int_{\overline{V}}\delta\bar{g}\,\mathrm{d}\overline{V}+\int_{\overline{V}}\bar{g}^{em}\delta u_{i,i}\,\mathrm{d}\overline{V}$. If we use a general expression $g(\bar{\varepsilon}_{IJ},\overline{E}_I,\overline{H}_I)$ instead of Eq. (7) we can discuss more general cases.

3.2. Alternative forms of the standard variational principles

(1) *First alternative forms of the standard variational principle*
From Eqs. (17)–(21) it is found that if we use $\overline{\mathbf{S}}$ instead of $\mathbf{S}$, $\overline{\mathbf{S}}^{\mathbf{env}}$ instead of $\mathbf{S}^{env}$ in the governing equations then the form of governing equation system of the physical nonlinear electromagnetic materials is just the same as that in the physical linear electromagnetic problem. Therefore we get a simpler principle-the first alternative form of the variational principle

$$\begin{aligned}&\delta\widehat{\Pi}=\delta\widehat{\Pi}_1+\delta\widehat{\Pi}_2-\delta\overline{W}^{int*}=0\\ &\delta\widehat{\Pi}_1=\int_V\delta\hat{g}\,\mathrm{d}V-\delta\overline{W}^*,\quad \delta\widehat{\Pi}_2=\int_{V^{env}}\delta\hat{g}^{env}\,\mathrm{d}V-\delta\overline{W}^{*env}\\ &\delta\hat{g}=\overline{S}_{ij}\delta u_{i,j}+\overline{D}_i\delta\varphi_{,i}+\overline{B}_i\delta\psi_{,i}\\ &\delta\hat{g}^{env}=\overline{S}^{env}_{ij}\delta u^{env}_{i,j}+\overline{D}^{env}_i\delta\varphi^{env}_{,i}+\overline{B}^{env}_i\delta\psi^{env}_{,i}\end{aligned}\tag{22}$$

From the mathematical view the variations of δu, $\delta\varphi$ and $\delta\psi$ are all local variations or completely independent, i.e. the migratory variations $\delta_u\varphi$ and $\delta_u\psi$ produced by $\delta\mathbf{u}$ are not needed. δW^*, δW^{env*} and δW^{int*} are still expressed by Eq. (11). It is also noted that in the magneto-elastic analysis Bustamante et al. (2008) used the analogous theory.

(2) *Second alternative forms of the standard variational principle*
If we assume that the body electromagnetic force $\bar{f}^{em}_k$ and surface electromagnetic force $\overline{T}^{em}_k$ in the media are

$$\bar{f}^{em}_k=\left(X_{L,k}\bar{\sigma}^M_{JL}\right)_{,J},\quad \overline{T}^{em}_k=X_{L,k}\bar{\sigma}^M_{JL}n_J\tag{23}$$

and the similar expressions for the environment. The second alternative form of the variational principle is

$$\begin{aligned}&\delta\widetilde{\Pi}=\delta\widetilde{\Pi}_1+\delta\widetilde{\Pi}_2-\delta W^{int*}=0\\ &\delta\widetilde{\Pi}_1=\int_{\overline{V}}\delta\bar{g}\,\mathrm{d}\overline{V}-\delta\widetilde{W}^*,\quad \delta\widetilde{\Pi}_2=\int_{\overline{V}^{env}}\delta\bar{g}^{env}\,\mathrm{d}\overline{V}-\delta\widetilde{W}^{env*}\\ &\delta\widetilde{W}^*=\int_{\overline{V}}(\bar{f}_k+\bar{f}^{em}_k-\bar{\rho}\ddot{u}_k)\delta u_k\,\mathrm{d}\overline{V}-\int_{\overline{V}}\bar{\rho}_e\delta\varphi\,\mathrm{d}\overline{V}\\ &\qquad+\int_{\bar{a}_\sigma}(\overline{T}^*_k-\overline{T}^{em}_k)\delta u_k\,\mathrm{d}\bar{a}-\int_{\bar{a}_q}\bar{\sigma}^*\delta\varphi\,\mathrm{d}\bar{a}+\int_{\bar{a}^{env}_\mu}\overline{B}^*_I\bar{n}_I\delta\psi\,\mathrm{d}\bar{a}\\ &\delta W^{*env}=\int_{\overline{V}^{env}}(\bar{f}^{env}_k+\bar{f}^{envem}_k-\bar{\rho}\ddot{u}^{env}_k)\delta u^{env}_k\,\mathrm{d}\overline{V}\\ &\qquad-\int_{\overline{V}^{env}}\bar{\rho}^{env}_e\delta\varphi^{env}\,\mathrm{d}\overline{V}+\int_{\bar{a}^{env}_\sigma}(\overline{T}^{env*}_k-\overline{T}^{envem}_k)\delta u^{env}_k\,\mathrm{d}\bar{a}\\ &\qquad-\int_{\bar{a}^{env}_q}\bar{\sigma}^{env*}\delta\varphi^{env}\,\mathrm{d}\bar{a}+\int_{\bar{a}^{env}_\mu}\overline{B}^{env*}_I\bar{n}_I\delta\psi^{env}\,\mathrm{d}\bar{a}\\ &\delta W^{int*}=\int_{\bar{a}^{int}}(\overline{T}^{int*}_k+\overline{T}^{envem}_k-\overline{T}^{em}_k)\delta u_k\,\mathrm{d}\bar{a}-\int_{\bar{a}^{int}}\sigma^{int*}\delta\varphi\,\mathrm{d}\bar{a}\\ &\qquad+\int_{\bar{a}^{int}_\mu}\overline{B}^{int*}_I\bar{n}_I\delta\psi^{env}\,\mathrm{d}\bar{a}\end{aligned}\tag{24}$$

From the mathematical view the variations of δu, $\delta\varphi$ and $\delta\psi$ are also completely independent, i.e. it is also not needed to consider the migratory variation $\delta_u \varphi$ and $\delta_u \psi$ produced by $\delta \mathbf{u}$. Eq. (24) is the original local form of Eq. (1), where the electromagnetic force is the external force calculated by the energy method prior. The governing equations got from this variational principle are

$$\begin{aligned}
&S_{ij,i} + (\bar{f}_j + \bar{f}_j^{em}) = \bar{\rho}\ddot{u}_j; \quad \bar{D}_{i,i} = \bar{\rho}_e; \quad \bar{B}_{i,i} = 0; \quad \text{in } \bar{V} \\
&S_{ij}\bar{n}_j = \bar{T}_j^* - \bar{T}_j^{em}, \quad \text{on } \bar{a}_\sigma; \quad \bar{D}_i\bar{n}_i = -\bar{\sigma}^*, \quad \text{on } \bar{a}_q; \\
&\bar{B}_i\bar{n}_i = \bar{B}_i^*\bar{n}_i, \quad \text{on } \bar{a}_\mu S_{ij,j}^{env} + \left(\bar{f}_i^{env} + \bar{f}_j^{envem}\right) = \bar{\rho}^{env}\ddot{u}_i^{env}; \\
&\bar{D}_{i,i}^{env} = \bar{\rho}_e^{env}; \quad \bar{B}_{i,i}^{env} = 0; \quad \text{in } \bar{V}^{env} \\
&S_{ij}^{env}\bar{n}_j^{env} = \left(\bar{T}_i^{env*} - \bar{T}_i^{envem}\right), \quad \text{on } \bar{a}_\sigma^{env}; \\
&\bar{D}_i^{env}\bar{n}_i^{env} = -\bar{\sigma}^{env*}, \quad \text{on } \bar{a}_q^{env}; \quad \bar{B}_i^{env}\bar{n}_i^{env} = \bar{B}_i^{env*}\bar{n}_i, \quad \text{on } \bar{a}_\mu^{env} \\
&\left(S_{kl} - S_{kl}^{env}\right)\bar{n}_l = \bar{T}_k^{int*} + \bar{T}_k^{envem} - \bar{T}_k^{em}, \\
&\bar{D}_k\bar{n}_k - \bar{D}_k^{env}\bar{n}_k = -\bar{\sigma}^{int*}, \bar{B}_k\bar{n}_k - \bar{B}_k^{env}\bar{n}_k = \bar{B}_k^{int*}\bar{n}_k; \quad \text{on } \bar{a}^{int}
\end{aligned} \tag{25}$$

In many literatures (Pao, 1978; Maugin, 1988; Moon, 1984) the electromagnetic force vector is included in governing equations. In their papers the electromagnetic force vector is derived from other methods different with the variational method and in different literatures $\bar{f}_j^{em}$ and $\bar{T}_j^{em}$ may be different.

(3) *The medium fully surrounded by the air under small deformation*
An important engineering problem is that the medium is fully surrounded by the air and the external magnetic field is applied on the boundary of air and far away from the medium. In air the mechanical stresses and mechanical energy approach zero, so only the electromagnetic field and electromagnetic energy should be considered. The analogous cases studied also by Toupin (1956) in electro-elastic media, Bustamante et al. (2008) in magneto-elastic media etc. When there is no body force for the above case and under the small deformation the physical variational formula (11) becomes

$$\begin{aligned}
&\delta\Pi = \delta\Pi_1 + \delta\Pi_2 - \delta W^{int*} = 0 \\
&\delta\Pi_1 = \delta\int_V g\,\mathrm{d}V - \delta W^*, \quad \delta\Pi_2 = \delta\int_{V^{env}} g^{env}\,\mathrm{d}V - \delta W^{env*} \\
&\delta W^* = -\int_V \rho\ddot{u}_k\delta u_k\,\mathrm{d}V - \int_V \rho_e\delta\varphi\,\mathrm{d}V \\
&\delta W^{env*} = -\int_{V^{env}} \rho_e^{env}\delta\varphi^{env}\,\mathrm{d}V + \int_{a_q^{env}} D_i^{env*}n_i\delta\varphi^{env}\,\mathrm{d}a \\
&\qquad + \int_{a_\mu^{env}} B_i^{env*}n_i\delta\psi^{env}da \\
&\delta W^{int*} = \int_{a_\sigma^{int}} T_k^{int*}\delta u_k da - \int_{a_q^{int}} \sigma^{int*}\delta\varphi da \\
&g = (1/2)C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - (1/2)\epsilon_{kl}E_kE_l - e_{kij}^eE_k\varepsilon_{ij} - (1/2)l_{ijkl}^eE_iE_j\varepsilon_{kl} \\
&\qquad - (1/2)\mu_{kl}H_kH_l - e_{kij}^mH_k\varepsilon_{ij} - (1/2)l_{ijkl}^mH_iH_j\varepsilon_{kl} \\
&g^{env} = -(1/2)\epsilon_{kl}^{env}E_k^{env}E_l^{env} - (1/2)\mu_{kl}^{env}H_k^{env}H_l^{env}
\end{aligned} \tag{26}$$

where the term $\Gamma_{kl}\varepsilon_{lk}$ have been neglected due to the strains are all small.

3.3. The effective electromagnetic force on the medium by the electromagnetic field

Comparing Eqs. (11) and (24) we find that the difference between them is that: In Eq. (11) we use the local variation and the migratory variation simultaneously, however in Eq. (24) we only use the local variation and the migratory variation is not appeared, but the work $\delta_u W^{em}$ done on the entire system by the electromagnetic force is introduced:

$$\begin{aligned}
\delta_u W^{em} &= \int_{\bar{V}} \bar{f}_k^{em}\delta u_k\,\mathrm{d}\bar{V} - \int_{\bar{a}_\sigma} \bar{T}_k^{em}\delta u_k\,\mathrm{d}\bar{a} + \int_{\bar{V}^{env}} \bar{f}_k^{envem}\delta u_k^{env}\,\mathrm{d}\bar{V} \\
&\quad - \int_{\bar{a}_\sigma^{env}} \bar{T}_k^{envem}\delta u_k^{env}\,\mathrm{d}\bar{a} + \int_{\bar{a}^{int}} \left(\bar{T}_k^{envem} - \bar{T}_k^{em}\right)\delta u_k\,\mathrm{d}\bar{a}
\end{aligned} \tag{27}$$

In Eq. (11) the part related to the migratory variations of electric and magnetic fields is

$$\begin{aligned}
&\int_{\bar{V}} \left(\bar{g}_{,\bar{\mathbf{E}}}\cdot\delta_u\bar{\mathbf{E}} + \bar{\mathbf{g}}_{,\bar{\mathbf{H}}}\cdot\delta_u\bar{\mathbf{H}}\right)\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{g}^{em}\delta u_{k,k}\,\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{\rho}_e\delta_u\varphi\,\mathrm{d}\bar{V} \\
&\quad + \int_{\bar{a}_q} \bar{\sigma}^*\delta_u\varphi\,\mathrm{d}\bar{a} - \int_{\bar{a}_\mu} \bar{B}_l^*\bar{n}_l\delta_u\psi\,\mathrm{d}\bar{a} \\
&\quad + \int_{\bar{V}^{env}} \left(\bar{g}_{,\bar{\mathbf{E}}}^{env}\cdot\delta_u\bar{\mathbf{E}}^{env} + \bar{g}_{,\bar{\mathbf{H}}}^{env}\cdot\delta_u\bar{\mathbf{H}}^{\mathbf{env}}\right)\mathrm{d}\bar{V} \\
&\quad + \int_{\bar{V}^{env}} \bar{g}^{emenv}\delta u_{i,i}^{env}\,\mathrm{d}\bar{V} + \int_{\bar{V}^{env}} \bar{\rho}_e^{env}\delta_u\varphi^{env}\,\mathrm{d}\bar{V} \\
&\quad + \int_{\bar{a}_q^{env}} \bar{\sigma}^{env*}\delta_u\varphi^{env}\,\mathrm{d}\bar{a} - \int_{\bar{a}_\mu^{env}} \bar{B}_l^{env*}\bar{n}_l\delta_u\psi^{env}\,\mathrm{d}\bar{a} \\
&\quad + \int_{\bar{a}^{int}} \bar{\sigma}^{int*}\delta_u\varphi\,\mathrm{d}\bar{a} - \int_{\bar{a}_\mu^{int}} \bar{B}_l^{int*}\bar{n}_l\delta_u\psi^{env}\,\mathrm{d}\bar{a}
\end{aligned} \tag{28}$$

Through some manipulation Eq. (28) is reduced to Eq. (27) and $\bar{f}_k^{em}$ and $\bar{T}_k^{em}$ are just the same as shown in Eq. (23) and the similar expressions for the environment. From the above discussion it is found that the effective or equivalent electromagnetic force applied on the medium can be obtained from the general energy migratory variational principle:

$$\begin{aligned}
\int_{\bar{V}} \bar{f}_k^{em}\delta u_k\,\mathrm{d}\bar{V} - \int_{\bar{a}} \bar{T}_k^{em}\delta u_k\,\mathrm{d}\bar{a} &= \int_{\bar{V}} \left(\bar{g}_{,\bar{\mathbf{E}}}\cdot\delta_u\bar{\mathbf{E}} + \bar{\mathbf{g}}_{,\bar{\mathbf{H}}}\cdot\delta_{\mathbf{u}}\bar{\mathbf{H}}\right)\mathrm{d}\bar{V} \\
&\quad + \int_{\bar{V}} \bar{g}^{em}\delta u_{k,k}\mathrm{d}\bar{V} \\
&\quad + \int_{\bar{V}} \bar{\rho}_e\delta_u\varphi\,\mathrm{d}\bar{V} + \int_{\bar{a}_q} \bar{\sigma}^*\delta_u\varphi\,\mathrm{d}\bar{a}
\end{aligned} \tag{29}$$

where $\bar{a} = \bar{a}_{\bar{\sigma}} + \bar{a}^{int}$. The similar expression is also valid for the environment. Eq. (29) may also be a definition of the effective electromagnetic force. The effective electromagnetic force applied on the interface is: $\bar{T}_k^{envem} - \bar{T}_k^{em} = X_{L,k}\bar{\sigma}_{JL}^{envM}n_J^{env} - X_{L,k}\bar{\sigma}_{JL}^{M}n_J$.

3.4. A note of the physical variational principle

In our previous paper (2008b) the variational principle is

$$\begin{aligned}
&\delta\bar{\Pi} = \delta\bar{\Pi}_1 + \delta\bar{\Pi}_2 - \delta\bar{W}^{int*} = 0 \\
&\delta\bar{\Pi}_1 = \delta\int_V g\,\mathrm{d}V - \delta\bar{W}^* = \int_V \delta g\,\mathrm{d}V + \int_V g\delta u_{i,i}\,\mathrm{d}V - \delta\bar{W}^* \\
&\quad = \int_{\bar{V}} \delta\bar{g}\,\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{g}\delta u_{i,i}\,\mathrm{d}\bar{V} - \delta\bar{W}^*
\end{aligned} \tag{a}$$

and other terms are omitted. In the present paper the variational principle is changed to

$$\begin{aligned}
&\delta\bar{\Pi} = \delta\bar{\Pi}_1 + \delta\bar{\Pi}_2 - \delta\bar{W}^{int*} = 0 \\
&\delta\bar{\Pi}_1 = \int_{\bar{V}} \delta\bar{g}\,\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{g}^{em}\delta u_{i,i}\,\mathrm{d}\bar{V} - \delta\bar{W}^*
\end{aligned} \tag{b}$$

and other terms are omitted. $\bar{g}$ in Eqs. (a) and (b) is shown in Eq. (9). If we use Eq. (a) we have

$$\begin{aligned}
\int_{\bar{V}} \delta\bar{g}\,\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{g}\delta u_{i,i}\,\mathrm{d}\bar{V} &= \int_{\bar{V}} \delta\bar{g}\,\mathrm{d}\bar{V} + \int_{\bar{V}} \bar{g}^{em}\delta u_{i,i}\,\mathrm{d}\bar{V} \\
&\quad + (1/2)\int_{\bar{V}} \bar{C}_{IJKL}\bar{\varepsilon}_{JI}\bar{\varepsilon}_{LK}\delta u_{i,i}\,\mathrm{d}\bar{V}
\end{aligned} \tag{c}$$

where

$$(1/2)\int_{\bar{V}}\overline{C}_{IJKL}\bar{\varepsilon}_{JI}\bar{\varepsilon}_{LK}\delta u_{i,i}\,\mathrm{d}\overline{V} = (1/2)\int_{\bar{a}}\overline{C}_{IJKL}\bar{\varepsilon}_{JI}\bar{\varepsilon}_{LK}n_i\delta u_i\,\mathrm{d}\bar{a} - (1/2)\int_{\bar{V}}\left(\overline{C}_{IJKL}\bar{\varepsilon}_{JI}\bar{\varepsilon}_{LK}\right)_{,i}\delta u_i\,\mathrm{d}\overline{V}$$

and the electromagnetic force (Maxwell stress) becomes to

$$\bar{\sigma}^M_{IL} = \overline{D}_I\overline{E}_L + \overline{B}_I\overline{H}_L - (1/2)\left(\overline{D}_N\overline{E}_N + \overline{B}_N\overline{H}_N + \overline{\Gamma}_{MN}\bar{\varepsilon}_{NM} - \overline{C}_{JKMN}\bar{\varepsilon}_{JK}\bar{\varepsilon}_{MN}\right)\delta_{IL} \tag{d}$$

So for the pure elastic situation the Maxwell stress is still equal to $\bar{\sigma}^M_{IL} = (1/2)\overline{C}_{JKMN}\bar{\varepsilon}_{JK}\bar{\varepsilon}_{MN}\delta_{IL}$ and does not reduce to zero. So that when Eq. (a) is used the physical variational principle cannot reduce to the usual elastic variational principle under finite deformation. If Eq. (b) (which is used in this paper) is used then in the pure elastic situation the Maxwell stress is equal to zero, so that the physical variational principle really reduces to the usual elastic variational principle under finite deformation.

Comparing with author's previous paper (2008b) the present physical variational principle is right in all possible deformation cases, so it is the exact and extended expression. In the present paper it is emphasized that for elasticity the original physical variational principle is a local variational principle and local variations would be used; but as an extension the effective electromagnetic force related to the migratory variation is also used. Using the energy conservation law from the virtual change of the sum of the electromagnetic energy and the couple energy produced by the migratory variation we can get the effective electromagnetic force. In this case the virtual variation of the volume should be considered because the energy conservation law is right for a virtual process from $\boldsymbol{u},\ \varphi,\ \psi$ to $\boldsymbol{u}+\delta\boldsymbol{u},\ \varphi+\delta\varphi,\ \psi+\delta\psi$.

3.5. A note of the Maxwell stress

From Eq. (15) it is known that under the finite deformation the Maxwell stress is related to the strain through $\overline{\Gamma}_{KL}\bar{\varepsilon}_{LK}$ in a material with piezoelectric or piezomagnetic behavior when the initial configuration is taken as a reference configuration. Because substantially the electromagnetic force is related to the distance between particles, the Maxwell stress relates to strains in finite deformation naturally. For the small deformation case the term containing $\Gamma_{lk}\varepsilon_{lk}$ can be neglected, so in the second order of precision the Maxwell stress is identical for all deformable and rigid body.

Pao (1978) pointed out that the uniqueness of the Maxwell stress is doubt, because from Maxwell electro-magnetic field equations different formulas of the Maxwell stress can be derived. Maugin (1988) considered that "... in order to avoid a fully arbitrary choice in the electromagnetic contributions to these equations, basically the so-called ponderomotive force and couple, and the corresponding rate of energy, we intend to arrive at sensible expressions for these contributions on the basis of a microscopic model ... the electron theory of Lorentz ...". Kankanala and Triantafyllidis (2004) listed three different formulas of Maxwell stresses. Bustamante et al. (2008) considered that "... the form of the Maxwell stress is dependent on which magnetic variable is taken to be independent and on the associated stress tensor." From the derivation of formulas in this paper, perhaps we can explain it in the following way: The electromagnetic Gibbs free energy of a material model is different in different literatures. Some theories did not consistent with the physical variational principle. In some previous literatures the boundary conditions did not considered. Authors usually directly discussed the rigid body, so the practical change of the distance between particles was not considered. If we discuss a deformable body (the rigid body can be considered as its limit case) with the isothermal electromagnetic Gibbs free energy as shown in Eq. (7) in this paper, then we can get a unique Maxwell stress vector for a definite material and under the finite deformation the Maxwell stress is related to the strains of materials. It is also noted that the Gibbs free energy of materials with internal dissipation mechanism or with body couple can not be expressed by Eq. (7), so the Maxwell stress of these materials are related to the special behavior of materials and is not studied here.

4. The first order theory of a thin elastic electromagnetic plate surrounded by air

4.1. Fundamental assumptions

The thin electro-magneto-elastic plate with thickness $2h$ surrounded by air under small deformation is often used in engineering. Here we give a first order theory of it as an application of the general theory. Let the origin of the coordinate system and the axes x_1, x_2 are located on the middle plane of the plate and the axis x_3 is downward along the thickness direction as in the usual literatures. It is also assumed that the plate bends downward. In the plate theory the role of the coordinate x_3 is different with x_1 and x_2 , so we discuss x_3 separately. As we know that the first order theory of a plate does not exactly satisfy the three dimensional elastic theory and the following main approximate assumptions are accepted in this paper:

(1) As in the classical plate theory it is assumed that $\sigma_{\alpha 3}$ $(\alpha = 1, 2)$ is one order less than $\sigma_{\alpha\beta}$, so part of the effect or the strains produced by $\sigma_{\alpha 3}$ can be neglected, i.e. $\varepsilon_{\alpha 3} \approx 0$. σ_{33} is fully neglected. In this section the subscript with Greek letter is taken $(1, 2)$ and the subscript with English letter is taken$(1, 2, 3)$.

(2) As in the classical plate theory the Kirchhoff assumption is adopted or it is assumed that the normal of the middle plane before deformation is still the normal of the middle plane after deformation (Timoshenko and Woinowsky-Krieger, 1959). Combing this assumption and $\varepsilon_{\alpha 3} \approx 0$, we get

$$\begin{aligned}
&u_k = u^0_k - x_3 u^0_{3,k}, \quad u^0_k = u^0_k(x_1, x_2, t), \quad u^0_{k,3} = 0, \quad (k = 1, 2, 3)\\
&\varepsilon_{\alpha\beta} = (1/2)\left(u^0_{\alpha,\beta} + u^0_{\beta,\alpha}\right) - x_3 u^0_{3,\alpha\beta}, \quad \varepsilon_{\alpha 3} = \varepsilon_{33} = 0
\end{aligned} \tag{33}$$

where we use $\mathbf{u}^0$ as the displacement vector on the middle surface, $\mathbf{u}$ is the displacement vector at any point in the plate. From Eq. (33) it is known that the displacement u_3 is all the same along the thickness direction.

(3) The electromagnetic field obeys the three dimensional theory and is not effected by the approximation of the displacements. But in order to consistent with the classical plate theory, the resultant electromagnetic forces introduced by the migratory variation of the electromagnetic energy and coupling energy are reduced to the middle plane s $(ds = dx_1 dx_2)$ or to the contour L of the middle plane.

4.2. Governing equations

When there is no body force in the medium surrounded by air the physical variational principle under small deformation is expressed in Eq. (26). According to the assumption (3) the electromagnetic field in the plate still obeys the three dimensional theory as that in previous sections, so we need not to consider it. But the electromagnetic force will be reduced to the middle plane s and its contour L. Therefore in the following part of this section we only discuss the mechanical motion equations and mechanical boundary conditions. Applying assumptions (1) and (2) in Eq. (26), we have

$$\begin{aligned}
\int_V \sigma_{kl}\delta u_{l,k}\,\mathrm{d}V &= \int_V \sigma_{kl}\delta u^0_{l,k}\,\mathrm{d}V - \int_V \sigma_{kl}\delta\left(x_{3,k}u^0_{3,l} + x_3 u^0_{3,lk}\right)\mathrm{d}V \\
&= \int_V \sigma_{\alpha\beta}\delta u^0_{\beta,\alpha}\,\mathrm{d}V - \int_V \sigma_{\alpha\beta}x_3\delta u^0_{3,\beta\alpha}\,\mathrm{d}V = \int_s \int_{-h}^{h} \sigma_{\alpha\beta}\,\mathrm{d}x_3\delta u^0_{\alpha,\beta}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad - \int_s \int_{-h}^{h} \sigma_{\alpha\beta}x_3\,\mathrm{d}x_3\delta u^0_{3,\alpha\beta}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \int_L N_{\alpha\beta}n_\beta\delta u^0_\alpha\,\mathrm{d}L - \int_s N_{\alpha\beta,\beta}\delta u^0_\alpha\,\mathrm{d}s - \int_L M_{\alpha\beta}n_\beta\delta u^0_{3,\alpha}\,\mathrm{d}L \\
&\quad + \int_L M_{\alpha\beta,\alpha}n_\beta\delta u^0_3\,\mathrm{d}L - \int_s M_{\alpha\beta,\beta\alpha}\delta u^0_3\,\mathrm{d}s \\
\int_V \rho\ddot{u}_k\delta u_k\,\mathrm{d}V &= \int_V \rho\left(\ddot{u}^0_k - x_3\ddot{u}^0_{3,k}\right)\delta\left(u^0_k - x_3u^0_{3,k}\right)\mathrm{d}V \approx \int_s \hat{\rho}\ddot{u}^0_k\delta u^0_k\,\mathrm{d}s
\end{aligned} \tag{34}$$

where

$$N_{\alpha\beta} = \int_{-h}^{h}\sigma_{\alpha\beta}\,\mathrm{d}x_3, \quad \mathrm{M}_{\alpha\beta} = \int_{-h}^{h}\sigma_{\alpha\beta}x_3\,\mathrm{d}x_3, \quad \hat{\rho} = \int_{-h}^{h}\rho\,\mathrm{d}x_3 \tag{35}$$

It is noted that $n_3 = 1,\ n_\alpha = 0$ on the middle planes, and $n_3 = 0$ on the lateral surface or on L. For the thin plate $h \to 0$ the electromagnetic forces on the interfaces between plate and air are:

$$\begin{aligned}
&\int_{a^{int}} \sigma^M_{ip}n_i\delta u_p da + \int_{a^{int}} \sigma^{env M}_{ip}n^{env}_i\delta u_p da \\
&= \int_L \int_{-h}^{h}\left(\sigma^M_{\alpha p} - \sigma^{env M}_{\alpha p}\right)dx_3 n_\alpha\delta u_p\,\mathrm{d}L \\
&\quad + \int_s \left(\sigma^M_{3p}\Big|^{upper}_{lower} - \sigma^{env M}_{3p}\Big|^{upper}_{lower}\right)n_3\delta u_p\,\mathrm{d}s \\
&= \int_L \left(N^M_{\alpha p} - N^{env M}_{\alpha p}\right)n_\alpha\delta u_p\,\mathrm{d}L + \int_s \left(p^M_i - p^{env M}_i\right)\delta u_i\,\mathrm{d}s \\
&= \int_L \left(N^M_{\alpha p} - N^{env M}_{\alpha p}\right)n_\alpha\delta u^0_p\,\mathrm{d}L - \int_L \left(M^M_{\alpha\beta} - M^{env M}_{\alpha\beta}\right)n_\alpha\delta u^0_{3,\beta}\,\mathrm{d}L \\
&\quad + \int_s \left(p^M_i - p^{env M}_i\right)\delta u^0_i\,\mathrm{d}s
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
&\sigma^M_{ip} = (D_iE_p + B_iH_p) - (1/2)(D_mE_m + B_mH_m)\delta_{ip} \\
&\sigma^{env M}_{ip} = \left(D^{env}_iE^{env}_\alpha + B^{env}_iH^{env}_\alpha\right) - (1/2)\left(D^{env}_mE^{env}_m + B^{env}_mH^{env}_m\right)\delta_{ip} \\
&N^M_{\alpha p} = \int_{-h}^{h}\sigma^M_{\alpha p}dx_3, \quad M^M_{\alpha p} = \int_{-h}^{h}\sigma^M_{\alpha p}x_3dx_3, \quad p^M_i = \sigma^M_{i3}\Big|^{upper}_{lower} \\
&N^{env M}_{\alpha p} = \int_{-h}^{h}\sigma^{env M}_{\alpha p}dx_3, \quad M^{env M}_{\alpha p} = \int_{-h}^{h}\sigma^{env M}_{\alpha p}x_3dx_3, \quad p^{env M}_i = \sigma^{env M}_{i3}\Big|^{upper}_{lower}
\end{aligned} \tag{37}$$

where $[A]^{upper}_{lower} = A_{upper} - A_{lower}$. The electromagnetic forces or the Maxwell stress in the plate are:

$$\int_V \sigma^M_{ip,i}\delta u_p\,\mathrm{d}V = \int_V \sigma^M_{i\alpha,i}\delta u_\alpha\,\mathrm{d}V + \int_V \sigma^M_{i3,i}\delta u_3\,\mathrm{d}V \tag{38a}$$

Because the plate is very thin we can assume that in the plate σ^M_{ip} is not related withx_3, or $\sigma^M_{ip} = \sigma^M_{ip}(x_1, x_2)$. So Eq. (38a) is reduced to

$$\begin{aligned}
\int_V \sigma^M_{ip,i}\delta u_p\,\mathrm{d}V &= \int_s \int_{-h}^{h}\sigma^M_{\alpha p,\alpha}\,\mathrm{d}x_3\delta u_p\,\mathrm{d}s \\
&= \int_s N^M_{\beta\alpha,\beta}\delta u^0_\alpha\,ds - \int_s M^M_{\beta\alpha,\beta}\delta u^0_{3,\alpha}\,ds + \int_s f^M_{\alpha,\alpha}\delta u^0_3\,ds
\end{aligned} \tag{38b}$$

where $f^M_\alpha = \int_{-h}^{h}\sigma^M_{\alpha 3}\,\mathrm{d}x_3$. On the interface the given surface traction is:

$$\begin{aligned}
&\int_{a^{int}_\sigma} T^{int*}_l\delta u_l da = \int_s p^{int*}_l\delta u^0_l ds + \int_{L_\sigma} P^{int*}_l\delta u^0_l\,\mathrm{d}L - \int_{L_\sigma} M^{int*}_l\delta u^0_{3,l}\,\mathrm{d}L \\
&p^{int*}_l = T^{int*}_l\Big|^{upper}_{lower}, ons; P^{int*}_l = \int_{-h}^{h}T^{int*}_l dx_3, \quad M^{int*}_l = \int_{-h}^{h}T^{int*}_l x_3 dx_3, \text{on } L
\end{aligned} \tag{39}$$

From Eqs. (34)-(39) we can get the mechanical part of the variational formula, but a modification should be carried out. According to the approximate assumption from Eq. (33) we have$\varepsilon_{\alpha 3} = 0$. But according to the equilibrium condition with the transverse force P^{int*}_3 we have $\sigma_{\alpha 3} \neq 0$. This contradiction is only solved by approximate method. From Eqs. (34)–(39) we can see that $\sigma_{\alpha 3}$ on the middle plane is approximately considered due to P^{int*}_3, but on the free boundary $\sigma_{\alpha 3}$ does not consider. So that a term $\int_L Q_\alpha n_\alpha\delta u_3\,\mathrm{d}L$ with $Q_\alpha = \int_{-h}^{h}\sigma_{\alpha 3}\,\mathrm{d}x_3$ on the (free) boundary we approximately add to the variational formula, i.e. we have

$$\begin{aligned}
&-\int_s \left[\left(N_{\alpha\beta} + N^M_{\alpha\beta}\right)_{,\beta} - \hat{\rho}\ddot{u}^0_\alpha + \left(p^{env M}_\alpha - p^M_\alpha\right) + p^{int*}_\alpha\right]\delta u^0_\alpha\,\mathrm{d}s \\
&+\int_L \left[\left(N_{\alpha\beta} - N^{env M}_{\alpha\beta} + N^M_{\alpha\beta}\right)n_\beta - P^{int*}_\alpha\right]\delta u^0_\alpha\,\mathrm{d}L \\
&-\int_s \left[\left(M_{\alpha\beta} + M^M_{\alpha\beta}\right)_{,\beta\alpha} + f^M_{\alpha,\alpha} - \hat{\rho}\ddot{u}^0_3 - \left(p^M_3 - p^{env M}_3\right) + p^{int*}_3\right]\delta u^0_3\,\mathrm{d}s \\
&+\int_L \left[\left(M_{\alpha\beta} + M^M_{\alpha\beta} - M^{env M}_{\alpha\beta}\right)_{,\alpha}n_\beta + Q_\beta n_\beta - P^{int*}_3 + \left(N^M_{\beta 3} - N^{env M}_{\beta 3}\right)n_\beta\right]\delta u^0_3\,\mathrm{d}L \\
&-\int_L \left[M_{\alpha\beta}n_\beta + \left(M^M_{\beta\alpha} - M^{env M}_{\beta\alpha}\right)n_\beta - M^{int*}_\alpha\right]\delta u^0_{3,\alpha}\,\mathrm{d}L = 0
\end{aligned} \tag{40}$$

From Eq. (40) the mechanical governing equations of the plane problem are:

$$\begin{aligned}
&\left(N_{\alpha\beta} + N^M_{\alpha\beta}\right)_{,\beta} + p^{env M}_\alpha - p^M_\alpha + p^{int*}_\alpha = \hat{\rho}\ddot{u}^0_\alpha; \quad \text{in } s \\
&\left(N_{\alpha\beta} + N^M_{\alpha\beta} - N^{env M}_{\alpha\beta}\right)n_\beta = P^{int*}_\alpha; \quad \text{on } L_\sigma
\end{aligned} \tag{41}$$

From Eq. (40) the mechanical governing field equation for the bending problem is:

$$\left(M_{\alpha\beta} + M^M_{\alpha\beta}\right)_{,\beta\alpha} + f^M_{\alpha,\alpha} + \left(p^{env M}_3 - p^M_3\right) + p^{int*}_3 = \hat{\rho}\ddot{u}^0_3; \quad \text{in } s \tag{42}$$

Usually the term $M^M_{\alpha\beta,\varepsilon\beta} + f^M_{\alpha,\alpha}$ is small comparing with other terms and can be neglected. The boundary conditions for the bending problem will be discussed more detail. Let (x_1, x_2) is coincide with (n, t) at a boundary point, where n and t denote the normal and tangent on the boundary L, respectively. The terms related to the boundary conditions in Eq. (40) are

$$\begin{aligned}
&\int_L \left[\left(M_{\alpha\beta} + M^M_{\alpha\beta} - M^{env M}_{\alpha\beta}\right)_{,\alpha}n_\beta + Q_\beta n_\beta - P^{int*}_3 + \left(N^M_{\beta 3} - N^{env M}_{\beta 3}\right)n_\beta\right]\delta u^0_3\,\mathrm{d}L \\
&-\int_L \left[M_{\alpha\beta}n_\beta + \left(M^M_{\beta\alpha} - M^{env M}_{\beta\alpha}\right)n_\beta - M^{int*}_\alpha\right]\delta u^0_{3,\alpha}\,\mathrm{d}L = 0; \quad (\alpha, \beta = n, t)
\end{aligned} \tag{43}$$

It seems that there are three boundary conditions in Eq. (43), but substantially there are two conditions in it. The usual three kinds of boundary conditions in plate theory can be discussed as follows:

(1) *Clamped side* According to the prior condition on the boundary given displacements of the variational principle it is needed from Eq. (43) that $\delta u^0_3 = \delta u^0_{3,\alpha} = 0$, or on the clamped side we have

$$u^0_3 = u^{0*}_3, u^0_{3,n} = u^{0*}_{3,n} \tag{44a}$$

where $u^0_{3,t} = u^{0*}_{3,t}$ is included in $u^0_3 = u^{0*}_3$. Where u^{0*}_3 and $u^{0*}_{3,n}$ are given values.

(2) *Hinged side* For the hinged side from Eq. (43) we have

$$u_3^0 = u_3^{0*}, \quad M_n + \left(M_n^M - M_n^{env M}\right) = M_n^{int*} \tag{44b}$$

where $M_n = M_{nn}$ and $u_{3,t}^0 = u_{3,t}^{0*}$ is included in $u_3^0 = u_3^{0*}$.

(3) *Free side* For the free side from Eq. (43) we get

$$\begin{aligned}&\left(M_{nt} + M_{nt}^M - M_{nt}^{env M}\right)_{,t} + Q_n + \left(N_{n3}^M - N_{n3}^{env M}\right) = P_3^{int*}\\ &M_n + \left(M_n^M - M_n^{env M}\right) = M_n^{int*}\end{aligned} \tag{44c}$$

It is noted that the usual three boundary conditions in the classical plate theory are derived from Eqs. (44a), (44b), (44c) without the Maxwell stress.

5. Examples

In this section we discuss simple examples to explain the application of the above theory.

5.1. Example 1: The force applied at the electrodes in a piezoelectric plate

A piezoelectric plate is located between upper and lower electrodes. Let axis x_2 is perpendicular to the electrode plate. The electric field $\mathbf{E} = E_2\mathbf{n} = \mathbf{i}_2 E_2$ is uniform due to that electrode plates are very large, where $\mathbf{n} = \mathbf{i}_2$ is the external normal of the upper surface of the plate and along the positive direction of the axis x_2. Let electrodes are free to external force, but on the interface an interfacial force $-\overline{T}_2^0$ is applied. In this case, we have $\varepsilon_{ij} = 0$, when $i \neq j$. Here the electrode plate is the environment of the electric medium. Inside the electrode we have $\mathbf{E} = \mathbf{0}$, so $E_2^{env} = 0, \ \bar{\sigma}_{2J}^{Menv} = 0$. According to Eq. (20), the boundary condition is $(\overline{S}_{IJ} - \overline{S}_{IJ}^{env})\bar{n}_I = T_2^{*int} = -\overline{T}_2^0$. Using Eqs. (15) and (17) the force per unit area on the upper electrode plate is

$$\begin{aligned}&\bar{\sigma}_{22}^M = \overline{D}_2\overline{E}_2 - (1/2)\left(\overline{D}_2 + \bar{e}_{2MN}^e\bar{\varepsilon}_{NM}\right)\overline{E}_2 = (1/2)\left(\overline{D}_2 - \bar{e}_{2MN}^e\bar{\varepsilon}_{NM}\right)\overline{E}_2\\ &T_2 = -\overline{S}_{22}^M - \overline{T}_2^0 = -X_{II,2}x_{2,II}\bar{\sigma}_{22}^M - \overline{T}_2^0 = \bar{\sigma}_{22}^M - \overline{T}_2^0\\ &\quad = -(1/2)\left(\overline{D}_2 - \bar{e}_{2MN}^e\bar{\varepsilon}_{NM}\right)\overline{E}_2 - \overline{T}_2^0\end{aligned} \tag{45}$$

From Eq. (45) the electromagnetic force is related to the strain when the initial configuration is taken as a reference configuration. For the case of small deformation we have $\bar{e}_{2MN}^e\bar{\varepsilon}_{NM} \approx 0$, so Eq. (45) is reduced to the usual formula in the electricity text book for dielectric. The force on the lower plate is opposite to that on the upper plate.

5.2. Example 2: An infinite ferromagnetic plate located in homogeneous magnetic fields

The vibration or buckling of a plate located in a homogeneous magnetic field is an important problem in the fusion power reactor engineering. Because there is no unified magnetic force formula in previous papers to solve the magnetoelastic instability in transverse magnetic field (Moon and Pao, 1968; Pao and Yeh, 1973; Hasanyan and Piliposyan, 2001) and the increase of natural frequency in an in-plane magnetic field in ferromagnetic plate (Takagi et al., 1993; Zhou and Miya, 1998), Zhou and Zheng (1997) proposed a special variational formula to solve it. However in the present paper a more unified, simple and rational electromagnetic force formula is given.

Let a ferromagnetic plate be located in a homogeneous magnetic field. Let the axis x_3 be perpendicular to the middle surface. Let the mechanical force is zero on the plate, so the magnetic force or the Maxwell stress is only the force acting on the plate. In the following part only the bending problem is discussed. According to Eq. (42) the governing equation of the bending problem of the plate is

$$\begin{aligned}&M_{\alpha\beta,\alpha\beta}^M + f_{\alpha,\alpha}^M + \left(p_3^{env M} - p_3^M\right), \quad p_i^M = \sigma_{i3}^M\big|_{lower}^{upper}\\ &\sigma_{i3}^M = (B_iH_3 - \frac{1}{2}B_mH_m\delta_{i3}), \sigma_{\alpha 3}^M = B_\alpha H_3, \sigma_{33}^M = (1/2)(B_3H_3 - \frac{1}{2}B_\alpha H_\alpha)\end{aligned} \tag{46}$$

where $B_i^{env} = \mu_0 H_i^{env}$ (in the air), $B_i = \tilde{\mu}H_i, \tilde{\mu} = \mu_0\mu_r = \mu_0(1+\chi)$, and μ_r is the relative permeability,χ is the magnetic susceptibility (in the plate).

The continuous conditions on the interfaces in this case are

$$H_\alpha^{env} = H_\alpha; B_3^{env} = B_3, or\ H_3^{env} = (\tilde{\mu}/\mu_0)H_3 \tag{47}$$

As pointed out in previous section the term $M_{\alpha\beta,\alpha\beta}^M + f_{\alpha,\alpha}^M$ can be neglected, so the magnetic force acting on the plate along $x_3$direction is reduced to

$$\begin{aligned}p_3^{env M} - p_3^M &= (1/2)\left[(B_3^{env}H_3^{env} - B_3H_3) - \left(B_\alpha^{env}H_\alpha^{env} - B_\alpha H_\alpha\right)\right]\Big|_{lower}^{upper}\\ &= -(1/2)\left[(1 - \tilde{\mu}/\mu_0)\tilde{\mu}H_3^2 - \left(\tilde{\mu}H_\alpha H_\alpha - \mu_0 H_\alpha H_\alpha\right)\right]\Big|_{lower}^{upper}\\ &= (1/2)\left[\mu_0\mu_r\chi H_3^2 + \mu_0\chi H_\alpha H_\alpha\right]\Big|_{lower}^{upper}\end{aligned} \tag{48}$$

When the external magnetic field is $\mathbf{H} = H\mathbf{i}_3$ Eq. (48) is slightly different with the formula of Eringen (1989) and identical with the formula given by Zhou and Zheng (1997). When the external magnetic field is $\mathbf{H} = H\mathbf{i}_1$ Eq. (48) is identical to the formula of Zhou and Miya (1998) in vale, but different with a sign. The reason may be that in Zhou and Miya's paper (1998) they only consider the variation of the magnetic energy by the volume change, and the terms containing $\delta_u\varphi$ and $\delta_u\psi$ did not considered. It is equivalent to consider the migratory terms $\int_{\overline{V}} \bar{g}^{em}\delta u_{i,i}\,d\overline{V}$ and $\int_{\overline{V}^{env}} \bar{g}^{emenv}\delta u_{i,i}^{env}\,d\overline{V}$ only in Eq. (11). The external magnetic force given in above two special cases can give the correct results in the vibration and buckling of a plate located in a homogeneous magnetic field (Zhou and Zheng, 1997).

6. Conclusions

In this paper we give an exact and extension form of the physical variational principle under finite deformation. It is shown that the original physical variational principle in elasticity is a local variational principle. However from the energy conservative principle the variations of the electromagnetic energy and coupling energy produced by the migratory variations of φ and ψ are equal to the work done by the effective electromagnetic force. In the energy conservative principle the virtual variation of the volume should be considered. In this paper it is also shown that the Maxwell stress is related to the finite strain for the media with piezoelectric or piezomagnetic behavior when the initial configuration is taken as a reference configuration. The thin plate theory including plane problem and bending problem is naturally derived from the general theory. The Maxwell stress is also naturally included in governing equations of a plate. This thin plate theory is fully complete for the nonlinear electro-magnetic media and the further applications are expected.

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